

SCHREIER THEORY OF TRACK CATEGORIES

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1. INTRODUCTION

This paper is a continuation of our study of non-abelian Baues-Wirsching cohomologies. In our previous paper [3], we defined second non-abelian cohomology $H^2(\mathcal{C}, D)$ of a small category \mathcal{C} with coefficients in a so-called centralised natural system D . We proved that $H^2(\mathcal{C}, D)$ classifies linear extensions of \mathcal{C} by D , generalising the corresponding result for abelian natural systems proved in [2].

For an abelian natural system D , the third cohomology classifies certain abelian track categories [1]. A track category is a 2-category where all 2-morphisms are isomorphisms. A track category is called abelian if for every 1-morphism f , the group $\text{Aut}(f)$ is abelian.

In a similar fashion to the above, we want to generalise this result for non-abelian track categories. In this paper we solve this problem for the following important case: Given categories \mathcal{K} and \mathcal{C} and a functor $\pi : \mathcal{K} \rightarrow \mathcal{C}$, which is identity on objects and surjective on morphisms, and G , a centralised natural system of groups on \mathcal{K} , we describe the equivalence classes of all track categories \mathcal{T} for which \mathcal{K} is the underlying category and \mathcal{C} is the homotopy category and $G_f = \text{Aut}(f)$.

2. PRELIMINARIES

2.1. Track categories. For a small groupoid \mathbf{G} , the set of its objects will be denoted by \mathbf{G}_0 and the set of morphisms by \mathbf{G}_1 . We have the source and target maps $\mathbf{G}_1 \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{t} \end{smallmatrix} \mathbf{G}_0$. A groupoid is called *abelian* if all automorphism groups are abelian.

Definition 1. A category enriched in groupoids \mathcal{T} , also termed track category, is a 2-category whose all 2-cells are invertible. It is thus a class of objects $\text{Ob}(\mathcal{T})$, a collection of groupoids $\mathcal{T}(A, B)$ for $A, B \in \text{Ob}(\mathcal{T})$ called hom-groupoids of \mathcal{T} , identities $1_A \in \mathcal{T}(A, A)_0$ and composition functors $\mathcal{T}(B, C) \times \mathcal{T}(A, B) \rightarrow \mathcal{T}(A, C)$ satisfying the usual equations of associativity and identity morphisms.

Thus $\mathcal{T}(A, B)_0$ denotes the set of objects of the groupoid $\mathcal{T}(A, B)$, while $\mathcal{T}(A, B)_1$ denotes the set morphisms of the groupoid $\mathcal{T}(A, B)$. Elements of $\mathcal{T}(A, B)_0$ we will call morphisms of \mathcal{T} , while elements of $\mathcal{T}(A, B)_1$ are called tracks of \mathcal{T} . The underlying category of \mathcal{T} is denoted by \mathcal{K} .

For $f, g \in \mathcal{T}(A, B)_1$ we shall write $f \simeq g$ (and say f is *homotopic* to g) if there exists a morphism α of \mathcal{T} from f to g . Occasionally this will also be denoted by $\alpha : f \simeq g$ or $\alpha : f \Rightarrow g$. Sometimes α is called a *homotopy* or a *track* from f to g . Homotopy is a natural equivalence relation on morphisms of \mathcal{K} and determines the homotopy category $\mathcal{C} = \mathcal{K} / \simeq$. Objects of \mathcal{C} are once again the objects in $\text{Ob}(\mathcal{T})$, while the morphisms of \mathcal{C} are the homotopy classes of morphisms in \mathcal{K} . We denote the canonical functor by π . So $\pi : \mathcal{K} \rightarrow \mathcal{C}$.

A morphism $g : B \rightarrow C$ in \mathcal{T} induces the functors

$$\begin{aligned} g_* : \mathcal{T}(A, B) &\rightarrow \mathcal{T}(A, C), & f &\mapsto gf, & \alpha &\mapsto g_*\alpha, \\ g^* : \mathcal{T}(C, D) &\rightarrow \mathcal{T}(B, D), & h &\mapsto hg, & \beta &\mapsto g^*\beta. \end{aligned}$$

These functors are restrictions of the composition functors. It follows from the definition that the following relations hold:

$$\text{TR 1} \quad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma),$$

- TR 2 $\alpha + 0 = \alpha = 0 + \alpha,$
- TR 3 $f^*(\alpha + \beta) = f^*(\alpha) + f^*(\beta),$
- TR 4 $g_*(\alpha + \beta) = g_*(\alpha) + g_*(\beta),$
- TR 5 $f^*(0) = 0 = g_*(0),$
- TR 6 $(ff_1)^* = f_1^* f^*, \quad 1^* = 1,$
- TR 7 $(gg_1)_* = g_* g_{1*}, \quad 1_* = 1,$
- TR 8 $g_* f^* = f^* g_*,$
- TR 9 $g_*(\alpha) + f_1^*(\alpha_1) = f^*(\alpha_1) + g_{1*}(\alpha).$

The following diagram explains the 1-morphisms and 2-morphisms in TR 9:

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{f_1} \end{array} B \begin{array}{c} \xrightarrow{g} \\ \xleftarrow{g_1} \end{array} C, \quad \alpha : f \Rightarrow f_1, \quad \alpha_1 : g \Rightarrow g_1.$$

The equality TR 9 holds in $\mathcal{T}(gf, g_1 f_1)$.

Definition 2. A track functor, or else 2-functor, $F : \mathcal{T} \rightarrow \mathcal{T}'$ between track categories is a groupoid enriched functor. Thus F assigns to each $A \in \text{Ob}(\mathcal{T})$ an object $F(A) \in \text{Ob}(\mathcal{T}')$, to each map $f : A \rightarrow B$ in \mathcal{T} a map $F(f) : F(A) \rightarrow F(B)$, and to each track $\alpha : f \Rightarrow g$ for $f, g : A \rightarrow B$ a track $F(\alpha) : F(f) \Rightarrow F(g)$ in a functorial way, i.e. so that one gets functors $F_{A,B} : \mathcal{T}(A, B) \rightarrow \mathcal{T}'(F(A), F(B))$. These assignments are compatible with identities and composition, or equivalently induce a functor $\mathcal{T}_1 \rightarrow \mathcal{T}'_1$, that is $F(1_A) = 1_{F(A)}$ for $A \in \text{Ob}(\mathcal{T})$, $F(fg) = F(f)F(g)$, and $F(\phi \times \psi) = F(\phi) \times (F\psi)$ for any $\phi : f \Rightarrow f'$, $\psi : g \Rightarrow g'$, $f, f' : A \rightarrow B$, $g, g' : B \rightarrow C$ in \mathcal{T} .

A track functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ is called a *weak equivalence* between track categories if the functors $\mathcal{T}(A, B) \rightarrow \mathcal{T}'(F(A), F(B))$ are equivalences of groupoids for all objects A, B of \mathcal{T} and each object A' of \mathcal{T}' is homotopy equivalent to some object of the form $F(A)$. Such a weak equivalence induces a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ between homotopy categories which is an equivalence of categories.

2.2. Natural systems. For a category \mathbf{I} , one denotes by \mathbf{FI} the *category of factorizations of \mathbf{I}* . Let us recall that objects of the category \mathbf{FI} are morphisms $f : i \rightarrow j$ of \mathbf{I} . A morphism from f to $f' : i' \rightarrow j'$ in \mathbf{FI} is a pair (g, h) , where $g : i' \rightarrow i$ and $h : j \rightarrow j'$ are morphisms in \mathbf{I} such that

$$f' = h \circ f \circ g.$$

In other words, the following diagram

$$\begin{array}{ccc} j & \xrightarrow{h} & j' \\ f \uparrow & & \uparrow f' \\ i & \xleftarrow{g} & k \end{array}$$

commutes. Composition in \mathbf{FI} is defined by

$$(g', h')(g, h) = (gg', h'h).$$

It is clear that

$$(g, h) = (g, \text{Id}_{j'}) (\text{Id}_i, h) = (\text{Id}_{i'}, h) (g, \text{Id}_j).$$

Let \mathbf{I} be a category. A *natural system* on \mathbf{I} with values in a category \mathbf{C} is a functor $D : \mathbf{FI} \rightarrow \mathbf{C}$. We denote the value of D on $f : i \rightarrow j$ by D_f or $D(f)$. If f is the identity

$\text{Id}_i : i \rightarrow i$ we write D_i instead of D_{Id_i} . We also write g^* and h_* instead of $D(g, \text{Id})$ and $D(\text{Id}, h)$. Then

$$D(g, h) = g^* h_* = h_* g^* : D_f \rightarrow D_{hfg}.$$

If \mathbf{C} is the category of sets; respectively groups, or abelian groups; we say that D is a natural system of sets; respectively groups, or abelian groups.

Let \mathbf{I} be a small category. A *centralised natural system of groups* is a natural system D of groups on \mathbf{I} such that for any arrows $i \xrightarrow{f} j \xrightarrow{g} k$ and elements $x \in D_f$, $y \in D_g$ one has the equality

$$g_*(x) + f^*(y) = f^*(y) + g_*(x)$$

in the group D_{gf} . By putting $i = j = k$ and $g = f = \text{id}_i$, it follows that for any object i the group D_i is an abelian group.

2.3. Natural system $\text{Aut}^{\mathcal{T}}$. Let \mathcal{T} be a track category. Recall that the underlying category \mathcal{K} has the same objects as \mathcal{T} , however the set of morphisms $\text{Hom}_{\mathcal{K}}(A, B)$ is the set of objects of the category $\mathcal{T}(A, B)$. For any morphism $f : A \rightarrow B$ of \mathcal{K} we let $\text{Aut}_f^{\mathcal{T}}$ be the collection of all automorphisms of f in the category $\mathcal{T}(A, B)$. Thus, this is the collection of all tracks $\alpha : f \Rightarrow f$. It follows from TR 1 and TR 2 that D_f is a group. Moreover, for any morphism $g : B \rightarrow C$, we have maps $g_* : \text{Aut}_f^{\mathcal{T}} \rightarrow \text{Aut}_{gf}^{\mathcal{T}}$ and $f^* : \text{Aut}_g^{\mathcal{T}} \rightarrow \text{Aut}_{gf}^{\mathcal{T}}$, which are group homomorphisms thanks to TR 3 – TR 5. Moreover, in this way one obtains a natural system $\text{Aut}^{\mathcal{T}}$ of groups on \mathcal{K} . This follows from the identities TR 6 – TR 8. We claim that $\text{Aut}^{\mathcal{T}}$ is centralised. To show this it suffices to put $f = f_1$ and $g = g_1$ in TR 9 to get:

$$g_*(\alpha) + f^*(\alpha_1) = f^*(\alpha_1) + g_*(\alpha).$$

3. COHOMOLOGY OF PRE-TRACK CATEGORIES

3.1. Pre-track categories. A pre-track category $(\pi : \mathcal{K} \rightarrow \mathcal{C}, G)$, or (π, G) for short, is the following data:

- (1) \mathcal{K} and \mathcal{C} are categories and π is a functor which is identity on objects and surjective on morphisms,
- (2) G is a centralised natural system of groups on \mathcal{K} .

Example 3. The main example for this is the following. Let \mathcal{T} be a track category and let \mathcal{C} be its homotopy category. Then $(\pi : \mathcal{K} \rightarrow \mathcal{C}, \text{Aut}^{\mathcal{T}})$ is a pre-track category.

So any track category gives rise to a pre-track category. The question is whether we can say something about the converse.

Definition 4. Let $(\pi : \mathcal{K} \rightarrow \mathcal{C}, G)$ be a pre-track category. A (π, G) -track category is a pair (\mathcal{T}, σ) where

- (1) $\mathcal{T} = (T\mathcal{K} \rightrightarrows \mathcal{K})$ is a track category with underlying category \mathcal{K} satisfying the property: for all $f, g \in \mathcal{K}(A, B)$ one has $\pi(f) = \pi(g)$ iff $\mathcal{T}(f, g) \neq \emptyset$.
- (2) σ is an isomorphism of natural systems on \mathcal{K}

$$\sigma : \text{Aut}^{\mathcal{T}} \rightarrow G.$$

Two (π, G) -track categories (\mathcal{T}, σ) and (\mathcal{T}', σ') are equivalent if there exists a track functor $F : \mathcal{T} \rightarrow \mathcal{T}'$ which is identity on objects and the following diagram commutes:

$$\begin{array}{ccc} \text{Aut}^{\mathcal{T}} & \xrightarrow{F} & \text{Aut}^{\mathcal{T}'} \\ & \searrow \sigma & \swarrow \sigma' \\ & G & \end{array}$$

One easily sees that if such an F exists, then it is an isomorphism of 2-categories. We let $\text{Tracks}(\pi, G)$ be the collection of equivalence classes of (π, G) -track extensions.

3.2. Second cohomology of pre-track categories. Given a pre-track category $(\pi : \mathcal{K} \rightarrow \mathcal{C}, G)$ we can also define the relative cohomology $H^2(\pi, G) := Z/\sim$ where Z is defined as follows.

Definition 5. Z is a collection of triples all (ξ, χ, ϕ) such that

- (1) ξ is a function which assigns to all triples of morphisms $f, g, h : i \rightarrow j$ of \mathcal{K} with $\pi(f) = \pi(g) = \pi(h)$, an element $\xi(f, g, h) \in G_f$,
- (2) χ is a function which assigns to each diagram $i \xrightarrow{f} j \xrightarrow{x} k$ with $\pi(f) = \pi(g)$ and $\pi(x) = \pi(y)$ an element $\chi(f, g|x, y) \in G_{xf}$,
- (3) ϕ is a function which assigns to each pair of morphisms f, g in \mathcal{K} with $\pi(f) = \pi(g)$ an isomorphism $\phi_{g,f} : G_g \rightarrow G_f$.

These functions must satisfy the following equations:

- (i) To simplify notation, first we define $m_*\phi_{b,a} : G_{mb} \rightarrow G_{ma}$ by

$$(m_*\phi_{b,a})(t) := -\chi(a, b|m, m) + \phi_{mb,ma}(t) + \chi(a, b|m, m)$$

where $a, b : i \rightarrow j$ and $m : j \rightarrow k$. Also, we define $b^*\phi_{n,m} : G_{nb} \rightarrow G_{mb}$ by

$$(b^*\phi_{n,m})(t) := -\chi(b, b|m, n) + \phi_{nb,mb}(t) + \chi(b, b|m, n),$$

where $b : i \rightarrow j$ and $m, n : j \rightarrow k$. Then we have the following equations:

- (a) For $f, g, h : i \rightarrow j$, $\phi_{g,f} \circ \phi_{h,g}(t) = \xi(f, g, h) + \phi_{h,f}(t) + \xi(f, g, h)$.
- (b) For $a, b : i \rightarrow j$, $m, n : j \rightarrow k$, $\beta \in G_b$ and $\nu \in G_n$, $m_*\phi_{b,a}(m_*\beta) = m_*(\phi_{b,a}(\beta))$, $a^*\phi_{n,m}(a^*\nu) = a^*(\phi_{n,m}(\nu))$,
- (c) and for $\mu \in G_m$ and $\alpha \in G_a$, $m_*\phi_{b,a}(b^*\mu) = a^*\mu$, $a^*\phi_{n,m}(n_*\alpha) = m_*\alpha$.
- (ii) For $f, g, h, e : i \rightarrow j$ with $\pi(f) = \pi(g) = \pi(h) = \pi(e)$, we have to have

$$\xi(f, g, e) + \phi_{g,f}\xi(g, h, e) = \xi(f, h, e) + \xi(f, g, h).$$

- (iii) For $x, y, z : i \rightarrow j$ with $\pi(x) = \pi(y) = \pi(z)$ and $a, b, c : j \rightarrow k$ with $\pi(a) = \pi(b) = \pi(c)$, we have to have

$$\xi(ax, by, cz) + \phi_{by,ax}(\chi(y, z|b, c)) + \chi(x, y|a, b) = \chi(x, z|a, c) + x^*\xi(a, b, c) + a_*\xi(x, y, z).$$

- (iv) For $x, y : i \rightarrow j$ with $\pi(x) = \pi(y)$, $a, b : j \rightarrow k$ with $\pi(a) = \pi(b)$ and $m, n : k \rightarrow l$ with $\pi(m) = \pi(n)$, we have to have

$$\chi(ax, by|m, n) + m_*\chi(x, y|a, b) = \chi(x, y|ma, nb) + x^*(\chi(a, b|m, n)).$$

Definition 6. We say two such triples (ξ, χ, ϕ) and (ξ', χ', ϕ') are equivalent, if there is an element $\zeta(f, g) \in G_f$ such that

- (1) $\zeta(f, h) + \xi(f, g, h) - \phi_{g,f}\zeta(g, h) - \zeta(f, g) = \xi'(f, g, h)$,
- (2) $\zeta(ax, by) + \chi(x, y|a, b) - x^*\zeta(a, b) - a_*\zeta(x, y) = \chi'(x, y|a, b)$, and
- (3) $\zeta(f, g) + \phi_{g,f}(t) - \zeta(f, g) = \phi'_{g,f}(t)$.

Theorem 7. There is a natural bijection

$$\text{Tracks}(\pi, G) \simeq H^2(\pi, G).$$

Proof. Let (\mathcal{T}, σ) be an element of $\text{Tracks}(\pi, G)$. Let $\pi(f) = \pi(g)$, so $\mathcal{T}(f, g) \neq \emptyset$. Choose a track $H_{f,g} \in \mathcal{T}(f, g)$ so that

$$H_{f,f} = 0, \quad H_{f,g} = -H_{g,f}.$$

Now take (f, g, h) in such a way that $\pi(f) = \pi(g) = \pi(h)$. Then there is a unique element $\xi(f, g, h) \in G_f$ such that

$$H_{f,g} + H_{g,h} = -\xi(f, g, h) + H_{f,h}.$$

Take $x, y \in K(i, j)$ and $a, b \in K(j, k)$ for which $\pi(x) = \pi(y)$ and $\pi(a) = \pi(b)$. Then there is a unique element $\chi(x, y|a, b) \in G_{ax}$ such that

$$a_*H_{x,y} + y^*H_{a,b} = -\chi(x, y|a, b) + H_{ax,by}.$$

For every f and g with $\pi(f) = \pi(g)$ we define an isomorphism $\phi_{g,f}(t) := H_{f,g} + t - H_{f,g}$, where $t \in T(g, g)$.

We claim that the triple (ξ, χ, ϕ) is an element of Z .

The Equations (i) of Definition 5 are straightforward to check. First, we check composition:

$$\begin{aligned}\phi_{g,f} \circ \phi_{h,g}(t) &= \phi_{g,f}(H_{g,h} + t - H_{g,h}) = H_{f,g} + H_{g,h} + t - H_{g,h} - H_{f,g} \\ &= -\xi(f, g, h) + \phi_{h,f}(t) + \xi(f, g, h).\end{aligned}$$

Next, we have

$$\begin{aligned}m_*(\phi_{b,a}(\beta)) &= m_*(H_{a,b} + \beta - H_{a,b}) = m_*H_{a,b} + m_*\beta - m_*H_{a,b} \\ &= (m_*\phi_{b,a})(m_*\beta).\end{aligned}$$

Similarly,

$$a^*(\phi_{n,m}(\nu)) = a^*(H_{m,n} + \nu - H_{m,n}) = (a^*\phi_{n,m})(a^*\nu).$$

The next equation

$$m_*\phi_{b,a}(b^*\mu) = m_*H_{a,b} + b^*\mu - m_*H_{a,b} = a^*\mu$$

follows from *TR9*, as shown by the diagram

$$i \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{b} \end{array} j \begin{array}{c} \xrightarrow{m} \\ \xleftarrow{m} \end{array} k, \quad H_{a,b} : a \Rightarrow b, \quad \mu : m \Rightarrow m.$$

The final equation is very similar to this.

To check (ii), we use the associativity property for the expression $H_{f,g} + H_{g,h} + H_{h,e}$ and the definition of the function ξ to obtain

$$\begin{aligned}(H_{f,g} + H_{g,h}) + H_{h,e} &= -\xi(f, g, h) + H_{f,h} + H_{h,e} = -\xi(f, g, h) - \xi(f, h, e) + H_{f,e} \\ H_{f,g} + (H_{g,h} + H_{h,e}) &= H_{f,g} - \xi(g, h, e) + H_{g,e} = H_{f,g} - \xi(g, h, e) - H_{f,g} - \xi(f, g, e) + H_{f,e}.\end{aligned}$$

So we have

$$\xi(f, h, e) + \xi(f, g, h) = \xi(f, g, e) + \phi_{g,f}(\xi(g, h, e)).$$

Let $x, y, z : i \rightarrow j$ and $a, b, c : j \rightarrow k$ be morphisms in K such that $\pi(x) = \pi(y) = \pi(z)$ and $\pi(a) = \pi(b) = \pi(c)$. Then we have

$$\begin{aligned}&a_*(H_{x,y} + H_{y,z}) + z^*(H_{a,b} + H_{b,c}) \\ &= a_*H_{x,y} + (a_*H_{y,z} + z^*H_{a,b}) + z^*H_{b,c} \\ &= a_*H_{x,y} + (y^*H_{a,b} + b_*H_{y,z}) + z^*H_{b,c} \\ &= -\chi(x, y|a, b) + H_{ax,by} - \chi(y, z|b, c) + H_{by,cz} \\ &= -\chi(x, y|a, b) + H_{ax,by} - \chi(y, z|b, c) - H_{ax,by} + H_{ax,by} + H_{by,cz} \\ &= -\chi(x, y|a, b) + H_{ax,by} - \chi(y, z|b, c) - H_{ax,by} - \xi(ax, by, cz) + H_{ax,cz}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&a_*(H_{x,y} + H_{y,z}) + z^*(H_{a,b} + H_{b,c}) \\ &= a_*(-\xi(x, y, z) + H_{x,z}) + z^*(-\xi(a, b, c) + H_{a,c}) \\ &= -a_*\xi(x, y, z) + a_*H_{x,z} - z^*\xi(a, b, c) - a_*H_{x,z} + a_*H_{x,z} + z^*H_{a,c} \\ &= -a_*\xi(x, y, z) + a_*H_{x,z} - z^*\xi(a, b, c) - a_*H_{x,z} - \chi(x, z|a, c) + H_{ax,cz} \\ &= -a_*\xi(x, y, z) - a_*\phi_{z,x}(z^*\xi(a, b, c)) - \chi(x, z|a, c) + H_{ax,cz} \\ &= -a_*\xi(x, y, z) - x^*\xi(a, b, c) - \chi(x, z|a, c) + H_{ax,cz} \quad \text{using Equation (i)}.\end{aligned}$$

We therefore obtain Equation (iii).

$$\xi(ax, by, cz) + \phi_{by,ax}(\chi(y, z|b, c)) + \chi(x, y|a, b) = \chi(x, z|a, c) + x^*\xi(a, b, c) + a_*\xi(x, y, z).$$

Let $x, y : i \rightarrow j$, $a, b : j \rightarrow k$ and $m, n : k \rightarrow l$ be morphisms in K such that $q(x) = q(y)$, $q(a) = q(b)$ and $q(m) = q(n)$. By the definition of χ we have

$$m_*H_{a,b} + b^*H_{m,n} = -\chi(a, b|m, n) + H_{ma,nb}$$

and

$$m_* a_* H_{x,y} + y^* H_{ma,nb} = -\chi(x, y|ma, nb) + H_{max,nby}.$$

Expressing $H_{ma,nb}$ from the first equation and substituting in the second, we obtain that the expression $-\chi(x, y|ma, nb) + H_{max,nby}$ is equal to

$$\begin{aligned} & m_* a_* H_{x,y} + y^* (\chi(a, b|m, n)) + y^* m_* H_{a,b} + y^* b^* H_{m,n} \\ &= m_* a_* H_{x,y} + y^* (\chi(a, b|m, n)) - m_* a_* H_{x,y} + m_* a_* H_{x,y} + y^* m_* H_{a,b} + y^* b^* H_{m,n} \\ &= m_* a_* H_{x,y} + y^* (\chi(a, b|m, n)) - m_* a_* H_{x,y} + m_* (a_* H_{x,y} + y^* H_{a,b}) + y^* b^* H_{m,n} \\ &= m_* a_* H_{x,y} + y^* (\chi(a, b|m, n)) - m_* a_* H_{x,y} - m_* \chi(x, y|a, b) + m_* H_{ax,by} + y^* b^* H_{m,n} \\ &= m_* a_* H_{x,y} + y^* (\chi(a, b|m, n)) - m_* a_* H_{x,y} - m_* \chi(x, y|a, b) - \chi(ax, by|m, n) + H_{max,nby} \\ &= x^* (\chi(a, b|m, n)) - m_* \chi(x, y|a, b) - \chi(ax, by|m, n) + H_{max,nby}, \text{ using Equation (i).} \end{aligned}$$

Therefore we have

$$\chi(ax, by|m, n) + m^* \chi(x, y|a, b) = \chi(x, y|ma, nb) + x^* (\chi(a, b|m, n)).$$

If we choose another $H'_{f,g} \in \mathcal{T}(f, g)$, there is a unique element $\zeta(f, g) \in G_f$ such that

$$\zeta(f, g) + H_{f,g} = H'_{f,g}.$$

We let ξ', χ' and ϕ' be the functions corresponding to H' . For all (f, g, h) with $\pi(f) = \pi(g) = \pi(h)$ we have

$$H'_{f,g} + H'_{g,h} = -\xi'(f, g, h) + H'_{f,h}.$$

Hence,

$$\begin{aligned} \zeta(f, g) + H_{f,g} + \zeta(g, h) + H_{g,h} &= \zeta(f, g) + H_{f,g} + \zeta(g, h) - H_{f,g} + H_{f,g} + H_{g,h} \\ &= -\xi'(f, g, h) + \zeta(f, h) + H_{f,h}. \end{aligned}$$

This gives

$$\zeta(f, h) + \xi(f, g, h) - \phi_{g,f} \zeta(g, h) - \zeta(f, g) = \xi'(f, g, h).$$

For $(x, y) \in \mathcal{K}(i, j)$ and $(a, b) \in \mathcal{K}(j, k)$ with $\pi(x) = \pi(y)$ and $\pi(a) = \pi(b)$, we have

$$a_* H'_{x,y} + y^* H'_{a,b} = -\chi'(x, y|a, b) + H'_{ax,by}.$$

So we get

$$\begin{aligned} a_* \zeta(x, y) + a_* H_{x,y} + y^* \zeta(a, b) + y^* H_{a,b} &= a_* \zeta(x, y) + a_* \phi_{y,x} (y^* \zeta(a, b)) - \chi(x, y|a, b) + H_{ax,by} \\ &= -\chi'(x, y|a, b) + \zeta(ax, by) + H_{ax,by}. \end{aligned}$$

Hence

$$\zeta(ax, by) + \chi(x, y|a, b) - x^* \zeta(a, b) - a_* \zeta(x, y) = \chi'(x, y|a, b).$$

We also have for all $(f, g) \in \mathcal{K}$ with $\pi(f) = \pi(g)$ and all $t \in G_f$

$$\zeta(f, g) + \phi_{g,f}(t) - \zeta(f, g) = \phi'_{g,f}(t).$$

The inverse map is constructed in the following way. Let $(\xi, \chi, \phi) \in H^2(\pi, G)$.

Let $\pi(f) = \pi(g)$. Then we define the set of tracks $T(f, g)$ to be the set $G_f \times f, g$. The track category structure is given by

$$(\alpha, f, g) + (\beta, g, h) = (\alpha + \phi_{g,f}(\beta) - \xi(f, g, h), f, h).$$

$$a_*(\alpha, f, g) = (a_*\alpha - \chi(f, g|a, a), af, ag),$$

$$b^*(\alpha, f, g) = (b^*\alpha - \chi(b, b|f, g), fb, gb).$$

Let us check that this indeed gives a track structure.

Now, to verify the relations $TR1 - TR9$: First, we check that the addition defined above is associative. We have

$$\begin{aligned} ((\alpha, f, g) + (\beta, g, h)) + (\gamma, h, e) &= (\alpha + \phi_{g,f}(\beta) - \xi(f, g, h), f, h) + (\gamma, h, e) \\ &= (\alpha + \phi_{g,f}(\beta) - \xi(f, g, h) + \phi_{h,f}(\gamma) - \xi(f, h, e), f, e) \\ &= (\alpha + \phi_{g,f}(\beta) + \phi_{g,f} \circ \phi_{h,g}(\gamma) - \xi(f, g, h) - \xi(f, h, e), f, e). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\alpha, f, g) + ((\beta, g, h) + (\gamma, h, e)) &= (\alpha, f, g) + (\beta) + \phi_{h,g}(\gamma) - \xi(g, h, e), g, e) \\ &= (\alpha + \phi_{g,f}(\beta) + \phi_{g,f} \circ \phi_{h,g}(\gamma) - \phi_{g,f}(\xi(g, h, e)) - \xi(f, g, e), f, e). \end{aligned}$$

Using Equation 5 (ii), we can see that we have equality.

Next, for *TR2* we have:

$$(\alpha, f, g) + (0, g, g) = (\alpha + \phi_{g,f}(0) - \xi(f, g, g), f, g) = (\alpha, f, g) = (0, f, f) + (\alpha, f, g)$$

. We also have that

$$\begin{aligned} m^*((\alpha, f, g) + (\beta, g, h)) &= m^*(\alpha + \phi_{g,f}(\beta) - \xi(f, g, h), f, h) \\ &= (m^*(\alpha + \phi_{g,f}(\beta) - \xi(f, g, h)) - \chi(m, m|f, h), fm, hm) \\ &= (m^*\alpha + m^*\phi_{g,f}(m^*\beta) - m^*\xi(f, g, h) - \chi(m, m|f, h), fm, hm) \\ &= (m^*\alpha - \chi(m, m|f, g) + \phi_{gm, fm}(m^*\beta) + \chi(m, m|f, g) - m^*\xi(f, g, h) - \chi(m, m|f, h), fm, hm). \end{aligned}$$

According to Equation 5 (iii), $-m^*\xi(f, g, h) - \chi(m, m|f, h) = -\chi(m, m|f, g) - \phi_{gm, fm}(\chi(m, m|g, h)) - \xi(fm, gm, hm)$. Therefore, we have

$$\begin{aligned} &= (m^*\alpha - \chi(m, m|f, g) + \phi_{gm, fm}(m^*\beta) - \phi_{gm, fm}(\chi(m, m|g, h)) - \xi(fm, gm, hm), fm, hm) \\ &= m^*(\alpha, f, g) + m^*(\beta, g, h). \end{aligned}$$

For *TR4*, we have very similarly:

$$\begin{aligned} x_*((\alpha, f, g) + (\beta, g, h)) &= x_*(\alpha + \phi_{g,f}(\beta) - \xi(f, g, h), f, h) \\ &= (x_*(\alpha + \phi_{g,f}(\beta) - \xi(f, g, h)) - \chi(f, h|x, x), xf, xh) \\ &= (x_*\alpha + x_*\phi_{g,f}(x_*\beta) - x_*\xi(f, g, h) - \chi(f, h|x, x), xf, xh) \\ &= (x_*\alpha - \chi(f, g|x, x) + \phi_{xg, xf}(x_*\beta) + \chi(f, g|x, x) - x_*\xi(f, g, h) - \chi(f, h|x, x), xf, xh). \end{aligned}$$

According to Equation 5 (iii), $-x_*\xi(f, g, h) - \chi(f, h|x, x) = -\chi(f, g|x, x) - \phi_{xg, xf}(\chi(g, h|x, x)) - \xi(xf, xg, xh)$. Therefore, we have

$$\begin{aligned} &= (x_*\alpha - \chi(f, g|x, x) + \phi_{xg, xf}(x_*\beta) - \phi_{xg, xf}(\chi(g, h|x, x)) - \xi(xf, xg, xh), xf, xh) \\ &= x_*(\alpha, f, g) + x_*(\beta, g, h). \end{aligned}$$

For *TR5*, we have

$$f^*(0, g, g) = (f^*0 - \chi(f, f|g, g), gf, gf) = (0, gf, gf) = (g^*0 - \chi(f, f|g, g), gf, gf) = g^*(0, f, f).$$

Next,

$$m^*n^*(\alpha, f, g) = m^*(n^*\alpha - \chi(n, n|f, g), fn, gn) = (m^*n^*\alpha - m^*\chi(n, n|f, g) - \chi(m, m|fn, gn), fnm, gnm).$$

Using Equation 5 (iv), we get that $-m^*\chi(n, n|f, g) - \chi(m, m|fn, gn) = -\chi(nm, nm|f, g)$, therefore giving us

$$= (n^*m^*\alpha - \chi(nm, nm|f, g), fnm, gnm) = (nm)^*(\alpha, f, g).$$

TR7 can be shown in a very similar manner. For *TR8*, we have

$$\begin{aligned} x_*m^*(\alpha, f, g) &= x_*(m^*\alpha - \chi(m, m|f, g), fm, gm) \\ &= (x_*m^*\alpha - x_*\chi(m, m|f, g) - \chi(fm, gm|x, x), xfm, xgm) \\ &= (x_*m^*\alpha - m^*\chi(f, g|x, x) - \chi(m, m|xf, xg), xfm, xgm), \\ &= m^*x_*(\alpha, f, g), \end{aligned}$$

due to Equation 5 (iv).

We also check $TR9$, the last equation:

$$\begin{aligned}
m_*(\alpha, a, b) + b^*(\mu, m, n) &= (m_*\alpha - \chi(a, b|m, m), ma, mb) + (b^*\mu - \chi(b, b|m, n), mb, nb) \\
&= (m_*\alpha - \chi(a, b|m, m) + \phi_{mb,ma}(b^*\mu - \chi(b, b|m, n)) - \xi(ma, mb, nb), ma, nb) \\
&= (m_*\alpha - \chi(a, b|m, m) + \chi(a, b|m, m) + m_*\phi_{b,a}(b^*\mu) - \chi(a, b|m, m) \\
&\quad - \phi_{mb,ma}(\chi(b, b|m, n)) - \xi(ma, mb, nb), ma, nb).
\end{aligned}$$

According to Equation 5 (iii), this equals to

$$\begin{aligned}
&= (m_*\alpha + m_*\phi_{b,a}(b^*\mu) - m_*\xi(a, b, b) - m_*\phi_{b,a}(b^*\xi(m, m, n)) - \chi(a, b|m, n), ma, nb) \\
&= (m_*\alpha + m_*\phi_{b,a}(b^*\mu) - \chi(a, b|m, n), ma, nb) = (m_*\alpha + a^*\mu - \chi(a, b|m, n), ma, nb).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
a^*(\mu, m, n) + n_*(\alpha, a, b) &= (a^*\mu - \chi(a, a|m, n), ma, na) + (n_*\alpha - \chi(a, b|n, n), na, nb) \\
&= (a^*\mu - \chi(a, a|m, n) + \phi_{na,ma}(n_*\alpha - \chi(a, b|n, n)) - \xi(ma, na, nb), ma, nb) \\
&= (a^*\mu + a^*\phi_{n,m}(n_*\alpha) - \chi(a, a|m, n) - \phi_{na,ma}(\chi(a, b|n, n)) - \xi(ma, na, nb), ma, nb).
\end{aligned}$$

According to Equation 5 (iii), this equals to

$$\begin{aligned}
&= (a^*\mu + a^*\phi_{n,m}(n_*\alpha) - m_*\xi(a, a, z) - m_*\phi_{b,a}(b^*\xi(m, n, n)) - \chi(a, b|m, n), ma, nb) \\
&= (a^*\mu + a^*\phi_{n,m}(n_*\alpha) - \chi(a, b|m, n), ma, nb) = (a^*\mu + m_*\alpha - \chi(a, b|m, n), ma, nb).
\end{aligned}$$

As G is a centralised natural system, we have $a^*\mu + m_*\alpha = m_*\alpha + a^*\mu$ in G_{ma} , therefore giving equality. \square

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